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Bound states and the Bekenstein bound

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Abstract: We explore the validity of the generalized Bekenstein bound, $S \leq \pi M a$. We define the entropy S as the logarithm of the number of states which have energy eigenvalue below M and are localized to a flat space region of width a. If boundary conditions that localize field modes are imposed by fiat, then the bound encounters well-known difficulties with negative Casimir energy and large species number, as well as novel problems arising only in the generalized form. In realistic systems, however, finite-size effects contribute additional energy. We study two different models for estimating such contributions. Our analysis suggests that the bound is both valid and nontrivial if interactions are properly included, so that the entropy S counts the bound states of interacting fields.

Keywords: Black Holes in String Theory, Models of Quantum Gravity.

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1. Introduction

Bekenstein's "universal entropy bound" [1] states that the entropy S of any weakly gravitating matter system obeys

$$S \leq 2\pi \frac{MR}{\hbar} \tag{1.1}$$

where M is the total gravitating mass of the matter and R is the radius of the smallest sphere capable of enclosing the system.[1]

Originally this bound arose from a gedankenexperiment by which the matter system is dropped into a large black hole. Classical general relativity implies that the black hole horizon area increases by $\Delta A \leq 8\pi GMR$ in this process. By the generalized second law of thermodynamics [5, 6, 2] (GSL), the total entropy visible to an outside observer will not decrease; hence, $S \leq \Delta A/4G\hbar$ and eq. (1.1) follows (see, e.g., ref. [7] for a review). We will not address here the controversial question concerning the extent to which quantum effects modify or invalidate this derivation from the GSL [8]–[12] (for reviews, see refs. [13, 14]).

The present paper studies aspects of a different question: Does the Bekenstein bound actually hold in nature? We are motivated to revisit this issue by a recent re-derivation [15] of Bekenstein's bound from the generalized covariant entropy bound [16] (GCEB), from which eq. (1.1) emerges in a somewhat stronger form:

$$S \leq \pi \frac{Ma}{\hbar}\,. \tag{1.2}$$

Here a is the smallest distance between any two parallel planes that bracket the system; a more precise definition is given in ref. [15]. Note that $a \leq 2R$, so that the "generalized Bekenstein bound", eq. (1.2), implies the original Bekenstein bound, eq. (1.1). (For example, for a rectangular box, a is the length of the shortest edge. For a sphere, $a = 2R$, and the two expressions agree.) Unlike the derivation of the Bekenstein bound from the GSL, the new derivation from the GCEB takes place in flat space, which excludes corrections from Unruh radiation.

The GCEB is a stronger version of the covariant entropy bound [17] (CEB), a conjecture concerning the entropy of matter in arbitrary spacetime regions. Compared to the GSL, the covariant bounds are less well-established hypotheses. However, they are more general (for example, they imply the GSL as a special case), and they are essential for putting the holographic principle [18, 19, 20] on a firm footing. If true, the covariant bounds will thus be of wider significance in quantum gravity.

This tempts us to regard the GCEB as the more fundamental conjecture, and to consider the Bekenstein bound a consequence of the GCEB. Tests of the GCEB generally require the computation of families of null geodesics, which can be complicated. The

[1] For an antecedent, see ref. [2]. We assume throughout that the system resides in Minkowski space. The Bekenstein bound has sometimes been applied to field systems on a curved background, and to strongly gravitating systems such as a black hole or a closed universe. With few exceptions [3], this is not sanctioned by its derivation, and we will not concern ourselves with violations [4] arising in this way.—We set $\hbar = k_{\mathrm{B}} = c = 1$, except in the introduction, where $\hbar$ is displayed explicitly. Newton's constant, G, is always displayed.

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Bekenstein bound[2] is more easily applied to examples, because the mass and size of many systems are known or easily determined. Thus the Bekenstein bound offers a simple way of testing the GCEB.

In refs. [16, 21], sufficient phenomenological conditions were identified under which the GCEB (and, directly or by extension, the Bekenstein bound) hold automatically. In order to determine its wider validity, our study of the Bekenstein bound will focus chiefly on examples in which the assumptions of refs. [16, 21] do not hold. Roughly speaking, this requires consideration of systems whose entropy is dominated by modes whose wavelength is of order the largest dimension.

To test eq. (1.2), one needs precise definitions of S, M, and a. Which type of entropy (canonical, microcanonical, entanglement, etc.) does the bound refer to? Different choices may agree in the high-temperature regime, but they are certainly inequivalent in those systems which are most likely to violate the Bekenstein bound.[3] What definition of energy should be used: the energy above the ground state, or the gravitating mass; an eigenvalue or an expectation value? Geometrically, the width a is sharply defined in the weak gravity limit. But what is the radius of a quantum state? Not surprisingly, analyses using different definitions come to different conclusions about the validity of the Bekenstein bound [1, 8, 9, 11],[24]–[38].

Our point of view is agnostic—we do not claim to know the correct formulation of Bekenstein's bound *a priori*. Rather, we would be content to find any reasonable definition of S, M, and a such that the Bekenstein bound becomes a precise, non-trivial, and empirically true statement. In the spirit of the covariant entropy bound, the general goal is to identify laws governing the information content of spacetime regions—patterns whose origin and implications may be of some significance.

Here we examine the validity of the Bekenstein bound with a microcanonical definition of entropy:

$$S \equiv \log \mathcal{N}(M)\,, \tag{1.3}$$

where $\mathcal{N}(M)$ the number of energy eigenstates with eigenvalue $E \leq M$. This definition is presented in detail in section 2. It is motivated empirically by additional difficulties that arise if canonical ensembles are considered [26], or if states other than energy eigenstates are admitted [37, 39].

We must also ensure that the systems we consider have finite width a. Initially, we enforce this by decree: we impose rigid boundary conditions restricting field modes to finite regions. We study mainly systems of scalar fields in finite cavities. In section 3, we verify that the bound is easily satisfied at high temperatures, where the canonical and microcanonical ensembles agree. We note that the bound becomes nearly saturated when the typical energy of quanta drops to the inverse size of the system.

[2]From here on, the term will refer to the generalized Bekenstein bound, eq. (1.2), unless specifically stated otherwise.

[3]The same systems are trivial in the context of the original covariant entropy bound [17], which is thus less sensitive to the precise definition of S than the GCEB. Indeed several proposals of cosmological entropy bounds were falsified by simple examples whose entropy is unambiguous (see, e.g., refs. [22, 23, 7]).

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In section 4, we construct situations in which the Bekenstein bound is apparently exceeded. Field systems with negative Casimir energy seem to violate eq. (1.1) by permitting its right hand side to become negative. Moreover, by proliferating the number of field species, the entropy of a cavity of fixed size and fixed energy can apparently be increased above (1.1). In addition to these two well-known difficulties we illustrate a new problem that occurs only in the tighter version (1.2) of the bound: Because the smallest width enters the right hand side, the entropy can apparently be made arbitrarily large by increasing the transverse area of a thin system at fixed energy and width (section 4.3).

In response to arguments raised against his bound, Bekenstein [27, 28, 38] has insisted that eq. (1.1) applies only to complete systems. Indeed, each of the above examples is an incomplete system, because fields were assumed to be localized to finite regions, but the matter causing this restriction was not included in the total energy.

Bekenstein has treated such examples by identifying missing parts and estimating their minimum mass [27, 28, 38] (and in some cases, also the additional entropy). This is satisfactory if the goal is to dispute that a given setup violates the bound. However, the procedure has been somewhat ad hoc, varying strongly depending on the specific example considered. It has not led to a clear definition of what does, in fact, constitute a "complete system", whose entropy and mass could be fully calculated in a single step.

In order to find an appropriate criterion, we will try to identify a single source of extra energy which contributes sufficiently to each and every one of the problematic examples so as to validate the bound. In sections 5 and 6, we devise two different models for completing the partial systems of sections 3 and 4. Each model implements one particular aspect that might be considered necessary for the spatial restriction of fields: confinement by interactions, and stability against pressure. We evaluate both models according to their ability to restore the validity of the Bekenstein bound without rendering it trivial.

In section 5, we note that the energy eigenstates of free fields are delocalized; to restrict to finite spatial width, interactions are essential. We model the confinement of the active fields to a region of with a by couplings to a stationary background potential. We estimate that the gradient energy of the background gives a lower bound on the energy we had failed to include in our previous analysis. We find that that this extra contribution was rightfully neglected in the unproblematic examples of section 3. However, in the apparent counterexamples of section 4, confinement effects are dominant. They significantly increase the total energy and restore the validity of Bekenstein's bound.

In section 6, we note that field modes restricted to a finite region exert pressure (or suction) on the region's boundary [27]. This pressure must be balanced by a tensile wall or "container", whose minimum energy we calculate as a function of its shape and the orthogonal pressure. This generalizes earlier estimates [27] to nonspherical systems. We find that this model is not satisfactory. Namely, the containment contribution by itself is not sufficient to eliminate counterexamples, although it does give large corrections where they are not needed (in the examples of section 3). Moreover, we argue that a container, and the resulting extra contribution to the energy, are in principle avoidable.

In section 7, we conclude that confinement by interactions constitutes an important physical mechanism by which the restriction to finite spatial width contributes to the total energy of a system, whereas the container model does not appear to capture an essential requirement. Our analysis shows that interactions play a crucial role and should be properly included from the start. Applying the bound to free fields obeying external boundary conditions robs the bound of its physical content. One must consider real interacting theories, which contain bound states of finite spatial extent. The entropy S is thus the logarithm of the number of bound states whose spatial width does not exceed a and whose energy eigenvalues satisfy $E \leq M$.

2. Modes, states, and entropy

In this section we describe in detail how the Bekenstein bound is applied to field systems confined to compact spatial regions. This involves the construction of a Fock space and the counting of admissible energy eigenstates. As a starting point, we enforce field localization by imposing external boundary conditions—an approach we will dismiss as unphysical in section 5.

Consider a free scalar field $\phi(t, x, y, z)$, of mass μ, occupying a stationary cavity of arbitrary size and shape. We will assume that ϕ respects Dirichlet boundary conditions ($\phi = 0$) on all cavity walls. The Fock space construction begins by expanding ϕ into energy eigenmodes,

$$\phi(x, y, z, t) = \sum_{\mathbf{k}} a_{\mathbf{k}}\, u_{\mathbf{k}}(x, y, z)\, e^{i\omega t} + \text{c.c.}\,, \tag{2.1}$$

whose spatial factor obeys the time-independent Schrödinger equation

$$\nabla_i \nabla^i u_{\mathbf{k}} = (\omega^2 - \mu^2) u_{\mathbf{k}}\,, \tag{2.2}$$

subject to the assumed Dirichlet boundary conditions.

Acting on the vacuum, the operator $a^\dagger_{\mathbf{k}}$ creates a one particle state of energy $\omega(\mathbf{k})$. More general energy eigenstates of the Fock space are constructed by acting with a variety of creation operators $a_{\mathbf{k}_1}$, $a_{\mathbf{k}_2}$, etc., one or multiple times. Thus, each energy eigenstate is labeled by a set of occupation numbers $\{N_{\mathbf{k}}\}$ listing the number of particles in each mode $\mathbf{k}$. The corresponding energy eigenvalues are

$$E_\phi(\{N_{\mathbf{k}}\}) = \sum_{\mathbf{k}} N_{\mathbf{k}} \omega(\mathbf{k})\,. \tag{2.3}$$

These states form a complete orthonormal basis of the Fock space.

We will generally consider a lagrangian with Q such scalars,[4] ϕ_q, $1 \leq q \leq Q$, which will be assumed not to interact mutually. Then eigenstates are characterized by occupation numbers $\{N^{(q)}_{\mathbf{k}}\}$ and energy eigenvalues

$$E_\phi(\{N^{(q)}_{\mathbf{k}}\}) = \sum_{\mathbf{k},q} N^{(q)}_{\mathbf{k}} \omega^{(q)}(\mathbf{k})\,. \tag{2.4}$$

[4]At high temperature we may think of Q more generally as an *effective* number of massless scalars, representing also massive scalars, as well as fermions and vector fields whose mass is negligible compared to the characteristic temperature. For example, in a cavity with conducting walls, for temperatures well below .5 MeV, we have $Q = 2$, corresponding to the polarization states of the photon.

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We define the entropy $S(M)$ microcanonically:

$$S(M) = \log \mathcal{N}(M) , \tag{2.5}$$

where $\mathcal{N}(M)$ is the number of energy eigenstates with energy $E \leq M$. This is equivalent to the statement that $S = \log \dim \mathcal{H}(M)$, where $\mathcal{H}(M)$ is the Hilbert space spanned by the energy eigenstates with $E \leq M$. Though we obtain S by counting pure states, note that a definition of S in terms of a density matrix $\hat{\rho}$ would be equivalent, as long as $\hat{\rho}$ is constructed entirely from the Hilbert space $\mathcal{H}(M)$. This is because Tr $\hat{\rho} \log \hat{\rho} \leq \log \mathcal{N}(M)$, with equality holding for the density matrix with equal weights for all states. However, definitions of S that permit the admixture of states with $E > M$ by superposition or in a density matrix are inequivalent and will be explored elsewhere [39]. In the present paper, M will be regarded as a sharp cutoff for energy eigenstates, not just as a cutoff for energy expectation values.

Other choices would have been conceivable, for example a canonical definition.[5] However, it is already of considerable interest to show that there exists *some* set of definitions under which Bekenstein's bound is well-defined, non-trivial, and not violated. Our choice is guided by the conceptual simplicity of counting pure states, and by the fact that Bekenstein's bound contains M (so it is natural to regard energy, not temperature, as the relevant macroscopic quantity to be fixed, and to work with energy eigenstates).

The Bekenstein bound does not contain Newton's constant, and it can be tested entirely within the non-gravitational setting of quantum field theory in flat space. However, in that context the bound might appear to be meaningless because there is no unique notion of absolute energy—one can always shift all values by a constant amount [26]. In order to define E we must recall the derivation of the Bekenstein bound from the focussing of light-rays [15] (before the zero gravity limit is taken): E is the gravitating mass of the matter system.

A central issue to be debated in this paper is the question of how E should be calculated. We will begin by defining E as the total energy of the ϕ field(s) in the cavity:

$$E = E_\phi + E_\mathrm{C} . \tag{2.6}$$

Here E_C is the Casimir energy, i.e., the energy of the vacuum state, which may be non-zero due to quantum effects. At a later stage (section 5), we will refine this definition of E and include additional contributions which we neglect for now.

The larger the field mass μ, the fewer states will be allowed below a fixed energy M, because some of the energy allotment is diverted into rest mass, which carries no entropy. Since we are interested in challenging Bekenstein's bound, we shall set $\mu = 0$ and consider only massless fields from here on.

[5]This would define S as the entropy of the density matrix of states at the temperature at which the energy expectation value is M. With this definition, apparent violations of Bekenstein's bound, in addition to those discussed in section 4, would occur at very low temperatures [26]. However, those violations would be resolved by the effects discussed in section 5 [38].

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3. Scalar fields in a cavity

In this section we verify that the Bekenstein bound, eq. (1.2), is satisfied by systems at high temperature, and that it becomes nearly saturated in the regime where the thermodynamic approximations begin to break down.

3.1 High temperature limit

In general, the microcanonical entropy, eq. (2.5), is difficult to calculate exactly. However, at high temperatures it agrees with the entropy of a corresponding canonical ensemble, which can be easily computed in a thermodynamic approximation.[6] For the thermodynamic description to be valid, the discreteness of the energy spectrum due to the finite size of the cavity should not be noticable. Thus we require that the characteristic wavelength of the radiation, $1/T$, be short compared to the length scales of the cavity. In particular,

$$T \gg \frac{1}{a} . \tag{3.1}$$

For the same reason, the temperature should also be much larger than the mass of the scalar field. This is not an important restriction: in any case we generally have massless fields in mind since they allow for the most entropy at a given energy.

Consider the canonical ensemble of the field(s) ϕ at the temperature T. With other parameters fixed, the energy expectation value, $\langle E\rangle(T)$, is a monotonic function of the temperature; we denote the inverse function by $T(E)$. Let $\mathcal{S}(T)$ be the entropy in the canonical ensemble. Then the microcanonical entropy of eq. (2.5) is given by

$$S(M) \approx \mathcal{S}[T(M)] \tag{3.2}$$

to an excellent approximation.

The expectation value for the total energy of the scalar field(s) is

$$\langle E\rangle(T) = \frac{\pi^2}{30} QVT^4 , \tag{3.3}$$

where V is the volume of the cavity. Recall that Q is the effective number of scalar degrees of freedom. The canonical entropy is

$$\mathcal{S}(T) = \frac{4}{3}\left(\frac{\pi^2}{30}\right) QVT^3 . \tag{3.4}$$

By inverting eq. (3.3) and using eq. (3.2) we find that the microcanonical entropy obeys

$$\frac{S}{\pi M a} \approx \frac{4}{3\pi T a} \ll 1 . \tag{3.5}$$

in the high temperature regime. Since $Ta \gg 1$ by assumption, the bound (1.2) holds comfortably. Except for this condition, the result is independent of the shape and volume of the cavity.

[6] In this regime, the assumptions of ref. [21] can be shown to be satisfied. Thus, we could simply appeal to that analysis. In the present context, however, explicit verification of the bound is straightforward and more instructive.

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Note that our result is also completely independent of the number of species, Q. This may seem surprising. After all, as Q grows, there are more ways to distribute the total energy into combinations of particles, so the entropy should increase without bound as $Q \to \infty$. However, this argument applies only if the total energy, or $\langle E \rangle$, is fixed. Here, we are holding the temperature at a fixed value, which must obey eq. (3.1). By eq. (3.3), increasing Q at fixed T leads to an increase in $\langle E \rangle$. Increasing Q at fixed $\langle E \rangle$, however, is accompanied by a decrease in T. The above calculation shows that violations of the bound cannot occur before T drops below the minimum value required for the validity of the canonical approximation, set by eq. (3.1).

3.2 Saturation limit

Having verified that the bound is always satisfied at temperatures much larger than the inverse size of a system, let us approach the opposite limit in which the thermodynamic approximation breaks down. If we extrapolate the high temperature expressions, (3.3) and (3.4), to the temperature $T \approx 1/a$, we find that the entropy and the right hand side of the bound are both of order Q, suggesting that the bound becomes saturated and perhaps even violated.

To resolve this issue, we can no longer rely on the canonical approximation to the microcanonical entropy. The discrepancy with the microcanonical entropy becomes a factor of order unity when $Ta \approx 1$. For energies around and below Q/a, the microcanonical entropy must be evaluated by explicit state counting. Before considering the extreme deformations (very large species number, or very thin cavity shapes) that lead to apparent difficulties, let us verify the validity of the bound for the generic case of a few species confined to a cavity whose various dimensions are roughly equal.

At low energy, the microcanonical entropy depends sensitively on the field content and the shape of the cavity, and on the boundary conditions employed. This forces us to select a specific (though representative) example, say a single ($Q = 1$) massless scalar field confined to a cubic cavity of side length a, with Dirichlet boundary conditions. This example, and many others, have been studied (in the context of the original Bekenstein bound) in ref. [29]. We reproduce here only the steps necessary to demonstrate explicitly how the entropy $S(M)$ is calculated and quote the main results.

The mode solutions are

$$\phi_{klm} = (2\omega)^{-1/2} \left(\frac{2}{a}\right)^{3/2} \sin\frac{k\pi x}{a} \sin\frac{l\pi y}{a} \sin\frac{m\pi z}{a} e^{i\omega t}\,, \tag{3.6}$$

with frequency

$$\omega = \frac{\pi}{a}\left(k^2 + l^2 + m^2\right)^{1/2}, \tag{3.7}$$

where k, l, and m run over the positive integers. We wish to verify that

$$S(M) \equiv \log \mathcal{N}(M) \leq \pi M a \tag{3.8}$$

for all values of M. We will set $E_{\mathrm{C}} = 0$ until section 4.1, where some implications of nonzero Casimir energy will be considered.

In calculations both of the entropy, and of the bound, only the product Ma enters; the overall scale of the system is not relevant. In the cubic example considered here, the smallest light-sheet width is a. In order to avoid cluttering our expressions with factors of π/a from eq. (3.7), we set

$$a = \pi \,. \tag{3.9}$$

Expressions for general a can be recovered by multiplying every energy by a factor a/π.

The only state with $0 \leq M < \sqrt{3}$ is the vacuum, so $S = 0$ and the bound is obeyed. For $M = \sqrt{3}$, a new state is allowed, namely $k = l = m = 1$ with occupation number 1; hence, $\mathcal{N}(\sqrt{3}) = 2$. Again the bound holds, since $S/\pi Ma = \log 2/\sqrt{3}\pi^2 < 1$. For $M = \sqrt{6}$, there are 3 additional states ($k = l = 1$, $m = 2$ and permutations, with occupation number one), so that $\mathcal{N} = 5$. However, $\log 5/\sqrt{6}\pi^2 \approx \frac{1}{11}$ is still less than unity. For larger M, one finds that the entropy increases to within an order of magnitude of saturating the bound, before asymptoting to the thermodynamic behavior

$$\frac{S}{\pi^2 M} \to \frac{4\pi}{3}\left(\frac{\pi}{30M}\right)^{1/4} . \tag{3.10}$$

More details are found in ref. [29]; see figure 1 therein.

To summarize, for a single scalar field ($Q = 1$) in a cube, the bound (1.2) is obeyed for any value of the total energy. The closest approach to saturation occurs at small values of M ($\approx 10/a$); at larger values, explicit state counting exhibits the onset of the asymptotic thermodynamic behavior $S \sim M^{3/4}$ described in section 3.1.

4. Challenges to the bound

In this section we describe various violations of the bound (1.2) which may occur with the definition (2.6) of the energy. In later sections we will argue that there are additional contributions to the energy which can salvage the bound.

4.1 Casimir problem

The finite size of the cavity gives rise to a shift of the vacuum energy density in its interior, whose integral is the Casimir energy, E_C. So far we have neglected this contribution. However, the Casimir energy is an inseparable part of the total gravitating energy E of any confined field, and our definition (2.6) calls for its inclusion. (There are further contributions arising from the enforcement of boundary conditions, which are not yet included in E and will be treated in later sections.) The Casimir energy, including its sign, depends sensitively on the field type, the detailed shape of the cavity, and the type of boundary conditions [40].

There are many well-known examples in which the Casimir energy is negative. For such cases it is immediately obvious that the bound can be violated. Simply choose M negative but of smaller magnitude than E_{C}. Then there is at least one state (the vacuum) with energy below M, and we have $S \geq 0$. Thus the entropy exceeds the bound, πMa, which is negative. Note that this difficulty occurs also in the original Bekenstein bound, eq. (1.1), in which the largest dimension of the system is used.

Though the above argument suffices to pose the Casimir problem in principle, a slightly more quantitative statement will later be needed to determine if we are able to resolve the problem. The Casimir energy is not generally known in closed form, but it can be calculated for many special cases and limits, including the rectangular cavities studied here. It will be sufficient to note that the energy of the vacuum state, in known examples, receives a Casimir correction which is of the general form

$$E_{\rm C} = \sum_{i=1}^{Q} \frac{\eta_i A}{a^3} . \tag{4.1}$$

Here a, the width of the system, is assumed to be the smallest dimension; A is the transverse area. Q is the effective number of scalar field degrees of freedom. The numbers η_i depend on the field type and the exact shape of the cavity. In known examples, the η_i are typically of order 10^{-1} or smaller and need not be positive [40]. Physically, this expression arises because the Casimir energy density is set by the smallest dimension of the system and hence is of the form η_i/a^4 for each species. Integration over the volume and summation over species yields eq. (4.1). This general estimate reduces to the results of ref. [38] for the case of an electromagnetic field confined to certain cavities.

4.2 Species problem

If the number of sufficiently light field species in a cavity, Q, is increased, the entropy will grow even though the total energy is held fixed. This is because for each particle in each state, one has an additional choice of species. Hence, the number of states with $N = \sum_{qklm} N_{klm}^{(q)}$ particles and energy $E_\phi \leq M$ increases by a factor of Q^N, while the right hand side of the bound remains unchanged. Thus, for any M large enough to allow at least the Q different one-particle states, there always exists some critical species number $Q_{\rm vio}$ such that the bound is violated for all $Q \geq Q_{\rm vio}$ [25]. This is known as the "species problem".[7]

The species problem is most acute when the number of quanta is small. Consider, for example, the cubic cavity introduced above and choose M so that at most a single particle of the lowest energy is admitted: $M = \sqrt{3}$. With $Q = 1$, we found in the previous section that $S = \log 2$, whereas the right hand side was somewhat larger, $\sqrt{3}\pi^2$. With Q noninteracting scalars, however, there are Q orthogonal one-particle states, plus the vacuum, all of which have $E \leq M$. Thus the entropy is given by

$$S = \log(1 + Q) \tag{4.2}$$

In this example the bound will be violated for $Q \geq e^{\sqrt{3}\pi^2} - 1 \approx 3 \times 10^7$.

Of course, there is no evidence that the actual number of species in Nature is so large; in this sense, the species problem is not manifest empirically. Indeed, one could regard the Bekenstein bound as a prediction of an upper limit on the number of field degrees of

[7] There are variants of this problem, for example a single species with extremely large spin. The analysis and resolution proceeds along similar lines.

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freedom [11]. We do not adopt this point of view here, because the model of section 5 suggests that when all contributions to the mass are included, the bound actually becomes more and more easily satisfied at large species number.

Note that the species problem does *not* occur in the covariant bound, since the length scale at which semi-classical gravity breaks down due to one-loop effects is $\ell_{\rm Pl}\sqrt{Q}$. The Bekenstein bound, however, does not contain Newton's constant. Arguments involving renormalization of G [41] cannot be advanced in its defense.

4.3 Transverse problem

Finally, we consider thin cavities, which give rise to a problem that is unique to the generalized Bekenstein bound (1.2), as opposed to eq. (1.1). A thin system is one with very unequal dimensions. The width, a, can be chosen to coincide with the smallest dimension [15]. A large transverse size allows orthogonal modes with nearly equal energy, which differ only in their large transverse wavelength. This gives rise to high entropy at low energy. Hence, for thin systems, the bound can be violated even for $Q = 1$. We call this the transverse problem.

There are two limits of interest, roughly the shapes of a pencil and of a pancake. We will consider these limits for a rectangular cavity of side lengths a, b, and c. Without loss of generality, we assume that

$$a \leq b \leq c\,. \tag{4.3}$$

(The special case $a = b = c$ was already considered above.) With Dirichlet boundary conditions, the mode solutions for a general rectangular cavity are

$$\phi_{klm} = \frac{2}{\sqrt{\omega abc}} \sin\frac{k\pi x}{a} \sin\frac{l\pi y}{b} \sin\frac{m\pi z}{c} e^{i\omega t}\,, \tag{4.4}$$

with frequency

$$\omega = \pi \left(\frac{k^2}{a^2} + \frac{l^2}{b^2} + \frac{m^2}{c^2}\right)^{1/2}, \tag{4.5}$$

where k, l, m are positive integers.

4.3.1 Pencil-shaped cavity

First, consider an elongated cavity with $a = b \ll c$. Instead of c, we find it convenient to use the dimensionless aspect ratio

$$\epsilon \equiv \frac{a}{c} \ll 1\,. \tag{4.6}$$

We choose the light-sheet that gives the shortest width, a, and we fix the overall scale of the cavity by choosing

$$a = \pi \tag{4.7}$$

as in the cubic example. Equation (4.5) simplifies to

$$\omega = \left(k^2 + l^2 + m^2\epsilon^2\right)^{1/2}. \tag{4.8}$$

For $0 \leq M < \sqrt{2+\epsilon^2}$, the bound is obeyed as the vacuum is the only state. For larger M, more and more single particle states with $k = l = 1$ and increasing m are admitted. In this mass range,

$$\sqrt{2+\epsilon^2} \leq M < \sqrt{5+\epsilon^2}\,, \tag{4.9}$$

the number of states is given exactly by

$$\mathcal{N}(M) = 1 + \frac{\sqrt{M^2-2}}{\epsilon}\,, \tag{4.10}$$

where it is understood that the integer part is taken.

Eq. (4.10) shows that for $M > \sqrt{2}$, the density of states can be made arbitrarily large by choosing the edge c to be much longer than a ($\epsilon \ll 1$). The entropy grows logarithmically with the transverse area A:

$$S \approx -\log\epsilon = \log\frac{A}{a^2}\,. \tag{4.11}$$

Thus, S can be increased arbitrarily while keeping M and a fixed. Hence, the generalized Bekenstein bound can be violated.

As M is increased above the range (4.9), states with higher (k, l) and multiparticle states become allowed. But eq. (4.10) still gives a lower bound on $\mathcal{N}(M)$. It follows that an aspect ratio of

$$\epsilon_{\rm vio} = M\exp(-\pi^2 M) \tag{4.12}$$

is sufficient to violate the bound for any $M \gg 1$. This means that one side of the box is exponentially longer than the other two sides.

For smaller values of M, the required aspect ratio does become practically feasible, though ϵ still has to be quite small to cause violation. To illustrate this, let us not fix M but find the thickest system (the largest ϵ) that allows violation for *some* choice of M. We take eq. (4.10) to be exact and assume that $\mathcal{N} \gg 1$. After the calculation we will demonstrate that this approximation is self-consistent.

From eq. (4.10), we see that the bound is violated if

$$\epsilon^2 < (M^2-2)\exp(-2\pi^2 M)\,. \tag{4.13}$$

The right hand side is maximal for

$$M(\epsilon_{\rm vio}^{\rm max}) = \frac{1}{2\pi^2} + \sqrt{2+\frac{1}{4\pi^4}} \approx 1.47\,. \tag{4.14}$$

Hence the largest aspect ratio leading to violation takes the value

$$\epsilon_{\rm vio}^{\rm max} \approx 2.01 \times 10^{-7}\,. \tag{4.15}$$

The critical mass $M(\epsilon_{\rm vio}^{\rm max})$ falls well below the value $\sqrt{5}$ at which eq. (4.10) becomes an underestimate. We also note that $\mathcal{N}(M)$ grows far less rapidly than $\exp(\pi^2 M)$ for $M > M(\epsilon_{\rm max})$ and that the bound is obeyed in the thermodynamic limit. Finally, we may verify that $\mathcal{N} \approx 1/\epsilon \gg 1$. Hence our above approximation is self-consistent and the bound cannot be violated for any values of ϵ exceeding $\epsilon_{\rm vio}^{\rm max}$. We conclude that pencil-shaped cavities whose length exceeds their diameter by a factor at least of order 10^7 admit violations of the bound at least for low values of M of order the inverse width of the cavity.

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4.3.2 Pancake-shaped cavity

Now consider a thin, pancake-shaped cavity with $a \ll b$ and $a \ll c$, and define the small aspect ratios $\delta \equiv a/b$ and $\epsilon \equiv a/c$. Taking $a = \pi$, eq. (4.5) simplifies to

$$\omega = \left(k^2 + l^2\delta^2 + m^2\epsilon^2\right)^{1/2} . \tag{4.16}$$

Again there is a gap, $M \approx 1$, above which densely spaced states become allowed. Below $M \approx 2$ all non-vacuum states are single particle states with $k = 1$. The number of states below M is given by

$$\mathcal{N}(M) = \frac{\pi(M^2 - 1)}{4\delta\epsilon} \tag{4.17}$$

to an approximation which is good for $\delta\epsilon \ll M^2 - 1 \leq 1$.

Again, we find that the entropy grows logarithmically with the transverse area A:

$$S \approx -\log\delta\epsilon = \log\frac{A}{a^2} . \tag{4.18}$$

Since the generalized Bekenstein bound is independent of A, it can be violated for sufficiently large A. This remains true even at larger values of the energy, $M > 2$, when the right hand side of eq. (4.17) becomes only a crude lower bound on $\mathcal{N}$.

It is interesting to ask "how thin" the cavity must be so that the bound can be violated for some M. The bound is violated if

$$\delta\epsilon < \frac{\pi}{4}(M^2 - 1)\exp(-\pi^2 M) . \tag{4.19}$$

The right hand side is maximal for $M = \pi^{-2} + (1 + \pi^{-4})^{1/2} \approx 1.11$. Hence the bound can be violated if

$$\delta\epsilon < 3.19 \times 10^{-6} . \tag{4.20}$$

In particular, for a square cavity ($\delta = \epsilon$), violations occur if the width-to-transverse-length ratio is at most

$$\epsilon_{\rm vio}^{\rm max} \approx 1.78 \times 10^{-3} . \tag{4.21}$$

4.3.3 Summary

We have considered thin rectangular cavities whose sidelengths obey $a \leq b \leq c$ and $bc \gg a^2$. We expect our conclusions to apply qualitatively for more general cavities with transverse area much larger than the smallest dimension squared ($A \gg a^2$).

Then eqs. (4.11) and (4.18) suggest that the entropy of any thin system at fixed low energy ($M \approx 1/a$) generally grows with the transverse area as

$$S \approx \log\frac{A}{a^2} , \tag{4.22}$$

whereas the right hand side of eq. (1.2) is independent of A and of order unity. For systems with dimensions of order $a^2/A \approx 10^{-6}$ or thinner, the entropy in eq. (4.22) therefore violates the bound.

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4.4 Comparing and combining problems

Among the three problem types we present here, two (species and Casimir) arose already in the context of the original Bekenstein bound (1.1), in which the circumferential diameter is used instead of the (smallest) width a. The different definition of the relevant length scale means that the original bound is immune to the transverse problem, which arises only for the generalized Bekenstein bound, eq. (1.2).

The different problems can exacerbate each other when they are combined. For example, one can trade high species number for thinness in concocting apparent counterexamples to the bound. For $Q > 1$, violations occur already at larger ("less thin") values of ϵ than those found in section 4.3.

Similarly, the Casimir problem can exacerbate the transverse problem. Suppose that $E_C < 0$ for a given combination of fields in the thin limit ($A \to \infty$ as a is held fixed). Take $Q = 1$ and recall the pencil-shaped cavity studied in section 4.3 and the conventions used there. The vacuum energy receives a Casimir shift of order η/ϵ. For the pancake shape, the shift is of order $\eta/\delta\epsilon$. In either case the shift can be made arbitrarily large, and the energy of many excited states will be negative, or positive but too small for the bound to be valid.

The species problem, like the thin-cavity problem, may also be exacerbated by Casimir energy. If the Casimir energy is negative for some field, then it can be made arbitrarily large in magnitude by including Q such fields. In such cases, $\mathcal{N}(M)$ grows with Q not only due to the greater degeneracy of energy levels, but also due to the negative contribution from E_{C}.

5. Confinement energy

So far, our analysis of field systems has assumed Dirichlet boundary conditions without including the matter required to enforce them. But a free field cannot be legislated to vanish outside a given region of space; physically, it must be forced to do so by couplings to other matter or to itself. If the Bekenstein bound applies only to complete systems, then our analysis so far will have fallen short.

For example, the electromagnetic field modes in a cavity will vanish on the boundary only if a sufficient number of electric charge carriers are available in the boundary material. Their density and coupling strength will also determine the depth of penetration of field modes into the surrounding conductor, and thus its minimum width. Bekenstein [38] recently showed that the extra energy contributed by such charge carriers suffices to restore the validity of the bound (1.1) in apparent counterexamples considered by Page [35, 36].[8]

The purpose of this section is to generalize this type of argument. A specific model is constructed in order to estimate a lower bound on the energy cost of confining fields to a region of width a. We then revisit our earlier examples and ask whether the extra energy resolves the apparent counterexamples found in section 4.

[8]One of Page's counterexamples [36] to eq. (1.1) is really a more complicated relative of the pencil problem of section 4.3.1. Our setup is simpler because it is designed to challenge only the stronger version of Bekenstein's bound, eq. (1.2).

5.1 Energy of a confining background field

5.1.1 A background field model of confinement

Let us begin by estimating what it takes to confine a single scalar field ϕ to a cavity (we will soon generalize this to multiple species). We model the confining matter by a turning on a static background field $\sigma(x, y, z)$ with standard kinetic term $(\nabla\sigma)^2$, which couples to the field ϕ through a term $\lambda\phi^2\sigma^2$. We must assume that the dimensionless coupling obeys

$$\lambda < 1\,, \tag{5.1}$$

since strong coupling leads to large corrections; in particular, one-loop contributions to the energy would not be under control. We will allow σ to be any function of the spatial coordinates, but we insist on adding the integrated gradient energy density of the σ field to the total energy:

$$E = E_\phi + E_{\mathrm{C}} + E_\sigma\,, \tag{5.2}$$

where

$$E_\sigma = \frac{1}{2}\int d^3x (\nabla_i\sigma)(\nabla^i\sigma)\,. \tag{5.3}$$

Implicit in this model is a distinction between an active field and a confining, non-dynamical background field. This is obviously somewhat artificial.[9] After all, dynamical fields are quite capable of confining themselves in interacting field theories, and in section 7, a revised definition of entropy will be given which refers to complete systems from the start. For now, our task is merely to estimate how much energy it takes to patch up examples that are physically incomplete field systems.

We begin with a set of field modes obeying sharp boundary conditions, as in sections 3 and 4. Next, we modify the system by couplings to a background field, enough to justify the spatial confinement of modes, but without significantly changing the number of states. Then we estimate the minimum energy contributed by the confining matter. Finally, we show that due to this extra energy, our naive conclusion that Bekenstein's bound is violated was not self-consistent.

The Fock space construction begins as usual by expanding ϕ into energy eigenmodes,

$$\phi(x, y, z, t) = \sum_{\mathbf{k}} a_{\mathbf{k}} u_{\mathbf{k}}(x, y, z) e^{i\omega t} + \mathrm{c.c.}\,, \tag{5.4}$$

which obey the Schrödinger equation

$$(\nabla_i\nabla^i + \lambda\sigma^2)u_{\mathbf{k}} = \omega^2 u_{\mathbf{k}}\,. \tag{5.5}$$

[9]Depending on the lagrangian, the model can be physically realized by solitons. Nonperturbative solutions generate a potential for perturbative states, some of which may be bound. Application of the Bekenstein bound in this context was investigated in ref. [32]. We emphasize, however, that our eventual definition of S in eq. (7.1) refers to any discrete eigenstate of the spectrum, i.e., not only to bound states arising in a background potential but also to those arising simply by perturbative interactions.—I wish to thank J. Bekenstein for pointing out ref. [32] to me.

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Acting on the vacuum, the operator $a^\dagger_{\mathbf{k}}$ creates a one particle state of energy $\omega(\mathbf{k})$. We call this particle *confined* to a region $\mathcal{V}$ if the support of the corresponding mode function is concentrated in that region, i.e., if

$$\int_{\mathcal{V}} d^3x\, u_{\mathbf{k}} u^*_{\mathbf{k}} \approx \int_{\mathbf{R^3}} d^3x\, u_{\mathbf{k}} u^*_{\mathbf{k}} \,. \tag{5.6}$$

More general energy eigenstates of the Fock space are constructed by acting with a variety of creation operators $a_{\mathbf{k}_1}$, $a_{\mathbf{k}_2}$, etc., one or multiple times. We call such a state *confined* to $\mathcal{V}$ if and only if all of its particles are confined to $\mathcal{V}$.

5.1.2 Estimating the confinement energy

To achieve sharp boundary conditions for the field ϕ, the background field σ would have to form an infinite potential well: inside $\mathcal{V}$, $\sigma = 0$; outside, $\sigma \to \infty$. This requires taking a limit in which σ changes by an infinite amount over zero spatial distance, so that E_σ will diverge. Hence, the cavities studied in sections 3 and section 4 would actually have infinite total energy.

In a cavity, all but a finite number of modes have $\omega > M$ and hence would not count towards the entropy S in any case. There is no point in confining such modes. Moreover, there is no need to suppress the amplitude of modes sharply; particles are considered well-localized to a region $\mathcal{V}$ even if there is a tiny probability to find them elsewhere.

Hence, consider instead a potential of finite height $\sigma_0 > 0$, such as a finite well with $\sigma = 0$ inside some compact region $\mathcal{V}$ and $\sigma = \sigma_0$ otherwise.[10] This potential can only bind ϕ-particles whose momenta have magnitude less than

$$k_0 = \sigma_0 \sqrt{\lambda} \tag{5.7}$$

in the rest frame of the background potential. Modes with larger momentum will not be bound. For rectangular $\mathcal{V}$, the bound modes will be roughly of the form of eq. (4.4), except that their amplitude will not turn off sharply at the potential step. Instead, the amplitude develops an exponential tail of width $(k_0^2 - k^2)^{-1/2}$.

A finite potential well still has infinite gradient energy. However, the modes change negligibly if the discontinuous jump of σ is slightly smeared. In order to reduce the gradient energy E_σ as much as possible, let us smear σ over the entire region $\mathcal{V}$, i.e., let σ rise continuously from 0 in the center of $\mathcal{V}$ to σ_0 at the boundary of $\mathcal{V}$; outside $\mathcal{V}$, we still take $\sigma = \sigma_0$, which costs no gradient energy at all.

We can now estimate a lower bound on the energy required to confine a single field species to a region of width a and transverse cross-section A. The gradient energy density is at least of order σ_0^2/a^2, since σ has to drop to 0 and rise back to σ_0 over a distance a. Integration over the volume Aa yields

$$E_\sigma \gtrsim \frac{A\sigma_0^2}{a} \gtrsim \frac{Ak_0^2}{a} \,, \tag{5.8}$$

where eqs. (5.1) and (5.7) have been used.

[10] We are choosing the region entering the Bekenstein bound to coincide with the potential well. This is not necessary, but it is the most interesting choice for the purposes of challenging the bound.

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5.1.3 Estimating the entropy

We now propose to test the generalized Bekenstein bound as follows. Pick a background field $\sigma(x, y, z)$ with finite gradient energy E_σ. For any $M \geq E_\sigma$ and any spatial region $\mathcal{V}$, let $\mathcal{N}(\sigma, \mathcal{V}; M)$ be the number of energy eigenstates in the Fock space which are confined to $\mathcal{V}$ and whose total energy E does not exceed M. Let a be the width of $\mathcal{V}$ as defined in the introduction (e.g., if $\mathcal{V}$ is a rectangular box, a can be chosen to be its smallest dimension). The entropy is defined microcanonically as

$$S(M) \equiv \log \mathcal{N}(\sigma, \mathcal{V}; M) \,. \tag{5.9}$$

The bound asserts that

$$S(M) \leq \pi M a \tag{5.10}$$

for all σ, $\mathcal{V}$, and M.

To calculate the entropy, one would need to find the eigenmodes of the smeared potential and count up all the confined Fock space states with total energy below $M - E_\sigma$. However, it is usually unnecessary to work out explicit mode solutions; a reasonable upper bound for the entropy can be estimated from the known solutions for an infinite well occupying the same region $\mathcal{V}$. Infinite well modes with momenta below k_0 will remain confined in a finite well with $\sigma = \sigma_0$ outside $\mathcal{V}$.[11] Under further deformation to a smooth potential valley, one would expect at most order one changes in the energy and size of most bound state modes. Also, since both quantities should increase under this deformation, we will be overestimating the entropy or underestimating the right hand side of the bound, which is permitted if we nevertheless find the inequality to be satisfied.

5.2 Examples

We will now revisit the various examples studied earlier, in order to include confinement in our analysis. We will find that confinement energy acts complementary to the field energy. In examples where the bound was already found to be easily satisfied, corrections from boundary effects are small, validating the earlier analysis. For examples where saturation of the bound was approached, the boundary effects lead to corrections of order one. Most importantly, for the problematic examples that appeared to violate the bound, we will find that confinement is actually the dominant source of energy, rendering our earlier analysis invalid.

5.2.1 High temperature limit

As our first example, let us revisit the high temperature limit of a scalar field in a cavity, studied in section 3.1. This is the regime

$$Ta \gg 1 \,, \tag{5.11}$$

[11] Bound states which are almost scattering states must also be excluded because they have large exponential tails outside $\mathcal{V}$ and so fail to satisfy eq. (5.6) (unless the characteristic length entering the bound is increased accordingly). E.g., one-dimensional modes with momentum k have tails of size $(k_0^2 - k^2)^{-1/2}$, which must not be much larger than the width of the potential valley; hence k must not be allowed to approach k_0 to closely.

in which thermodynamic approximations are valid. We set the number of species to one ($Q = 1$), postponing the analysis of $Q > 1$ to section 5.2.4.

In order to apply the general result for confinement energy, eq. (5.8), we need to know how large k_0 has to be. In the thermal regime, the typical ϕ-modes have energy and momenta of order T, so confinement is achieved by choosing $k_0 = T$. Since a is the shortest width, we have $A \geq a^2$, and eq. (5.8) implies

$$E_\sigma \gtrsim T^2 a\,. \tag{5.12}$$

As a function of T, the entropy is

$$S(T) \approx (Ta)^3 \tag{5.13}$$

whereas the right hand side of the generalized Bekenstein bound contains two terms

$$Ea \approx E_\phi a + E_\sigma a \approx (Ta)^4 + (Ta)^2 \tag{5.14}$$

by eqs. (3.3) and (5.12).

At leading order in the high temperature limit, we see that the ratio

$$\frac{S}{Ea} \approx \frac{1}{Ta} \tag{5.15}$$

is not affected by confinement. The Bekenstein bound is satisfied with the same comfortable factor (Ta) to spare that we already determined in section 3.1. This is a sensible result. If a system is much larger than the typical wavelengths it contains, then the contribution of boundaries to the energy should not have to be large.

5.2.2 Saturation limit

Next, let us turn to the limit in which saturation of the bound may be approached, as exemplified in section 3.2. This occurs for systems whose various dimensions are all roughly equal to a single length scale a. Examples include a cubic (or spherical) cavity of side length (or diameter) a confining a single massless scalar. Saturation is approached as the maximum total energy, M, is lowered to become of order $1/a$. Then the cavity is occupied by a small number of quanta whose typical wavelength is comparable to the cavity diameter, a. Since the entropy is of order one, the canonical ensemble cannot be used as an approximate substitute for the microcanonical counting of states we employ.

Recall that the total energy of the ϕ-states, in this limit, is comparable to the energy of an individual ϕ-particle, i.e.,

$$E_\phi \approx \frac{1}{a}\,. \tag{5.16}$$

Hence $E_\phi a$ is of order unity, like the entropy. The momenta of confined ϕ particles are at least $1/a$ by the uncertainty relation. Thus, eq. (5.8), with $A \approx a^2$, implies

$$E_\sigma \gtrsim \frac{1}{a}\,. \tag{5.17}$$

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Hence, the confinement contribution to the right hand side of the bound is of the same order as that of the active field ϕ. Compared to the analysis without attention to confinement, the bound is now satisfied with an extra factor $\gtrsim 1$ to spare.

We conclude that upon inclusion of confinement energy, systems with very few quanta can still come reasonably close to saturating the bound. In this limit, however, the boundary contribution begins to dominate over the field energy. This would appear to make it even more difficult to devise simple systems in which the bound is saturated exactly.

5.2.3 Resolving the Casimir problem

If the vacuum state had negative energy $E_{\rm C}$, the Bekenstein bound would be automatically violated for negative values of $M \geq E_{\rm C}$ (section 4.1). The calculation of Casimir energies is often carried out with sharp boundary conditions separating bound states inside the region $\mathcal{V}$ from the continuous spectrum outside. As we have discussed above, perfect confinement gives rise to infinite barrier energy. However, imperfect barriers also lead to Casimir energies, and eq. (4.1) retains its general form even with an ultraviolet cutoff. Hence, we will only assume that the background potential confines at least one particle to a region of width a.

As in the previous example, the uncertainty principle demands that the barrier height, k_0, must be greater than $1/a$. By eq. (5.8), the confining matter therefore has at least energy

$$E_\sigma \gtrsim \frac{1}{a}\,. \tag{5.18}$$

Comparison with eq. (4.1) shows that the Casimir energy is parametrically smaller in magnitude than the positive confinement energy [27]. This suggests that the total energy is always positive—a result which one expects for complete systems on physical grounds in any case [33].

5.2.4 Resolving the species problem

At fixed energy, the entropy of weakly interacting fields confined to a fixed compact region will increase monotonically with the number of field species. This constitutes the species problem (section 4.2). We will assume a region of regular shape, $A \approx a^2$, for simplicity.

If a type of particle is not confined to the given region, its modes do not contribute to the entropy. For example, one might build a box that confines photons and electrons, but not neutrinos; in that case, one would not include neutrino states in S. Hence, we must adapt our estimate of confinement energy, eq. (5.8), to the problem of confining multiple fields. We will focus on two simple methods which we expect to be representative, and which give the same answer.

One possibility is to couple each active field ϕ_q to a different background potential σ_q via coupling terms

$$\sum_q \lambda_q \phi_q^2 \sigma_q^2\,. \tag{5.19}$$

Since we must require that none of the couplings are strong ($\lambda_q < 1$ for all q), each background field σ_q will contribute gradient energy

$$E_{\sigma_q} \approx \int (\nabla\sigma_q)^2 \gtrsim \frac{1}{a}\,. \tag{5.20}$$

Hence the total confinement energy obeys

$$E_\sigma \gtrsim \frac{Q}{a}\,. \tag{5.21}$$

A second possibility is to couple several or all fields ϕ_q to the same background potential σ via terms $\lambda \sum_q \phi_q^2\sigma^2$. Then a potential σ that confines certain modes of ϕ_1 to a compact region will confine the corresponding modes of all other fields. Hence, it would appear that the confinement energy will be independent of the number of fields, Q. However, radiative corrections to the σ propagator will now be larger, $Q\lambda$, because Q fields are running in the loop. Hence, eq. (5.1) is no longer sufficient to bring such corrections under control. Instead, we must demand that

$$\lambda < \frac{1}{Q}\,. \tag{5.22}$$

This modifies our estimate for the confinement energy: eq. (5.8) must be replaced by

$$E_\sigma \gtrsim \frac{A\sigma_0^2}{a} \gtrsim \frac{QAk_0^2}{a}\,. \tag{5.23}$$

Compared to the previous paragraph, we hoped to save gradient energy by coupling all active fields to the same background field. But at the same time we gained gradient energy because the couplings are now restricted to be weaker, requiring a greater potential height σ_0. The result is the same in both cases: eq. (5.21).

In order to obtain apparent violations in section 4.2, the number of species had to be chosen very large: $Q \gg 1$. Since $E_\phi \approx 1/a$ in the mass range where the bound is most easily threatened, the total energy will be dominated by confinement effects for $Q \gg 1$, so that the right hand side of Bekenstein's bound is at least of order Q. This conservative estimate increases linearly with Q, while the entropy on the left hand side grows only logarithmically. Hence, the bound, far from being violated, actually becomes more comfortably satisfied when the number of species is increased in the background field model.

Physically, this result is easily understood as follows. Start with a well-localized system (say, a few atoms forming a small crystal) and increase the species number at fixed energy. If the new species all couple with the same strength as the original ones, then the system will develop strong 't Hooft couplings. It will be transformed by radiative corrections, rendering its size and entropy difficult to calculate and invalidating the naive argument for a violation of the bound. If, on the other hand, some of the new species are more weakly coupled than the original ones, so as to avoid strong 't Hooft coupling, then the system will delocalize. The more species, the weaker the required couplings, so that the system grows in size, and the bound increases more rapidly with species number than the entropy does.

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5.2.5 Resolving the transverse problem

Neglecting confinement, we noted in section 4.3 that the generalized (though not the original) Bekenstein bound is violated by "thin" systems, whose width a is much smaller than its transverse size. Choosing the maximum energy M to be of order $1/a$, one finds that only one-particle states are allowed, whose number grows as a power law with the transverse area. Hence, the entropy grows logarithmically with the transverse area; see eq. (4.22). Yet the right hand side of the bound is fixed and of order unity.

The estimate of the confinement energy is simple. We need only consider the contributions from the transverse surfaces (e.g., the capacitor plates) to eq. (5.8). This contribution dominates over the other boundaries of the system for several reasons: (i) the transverse boundaries have the largest area, A; (ii) they set the required barrier height σ_0, because $1/a$ is the largest momentum component; and (iii) they contribute the largest gradient energy density to $(\nabla\sigma)^2$ because a is the shortest dimension. Substituting $k_0 \approx 1/a$ in eq. (5.8), we find for the total confinement energy:

$$E_\sigma \gtrsim \frac{A}{a^3}\,. \tag{5.24}$$

The confinement contribution to the right hand side of the entropy bound is $E_\sigma a$. By the above inequality, this is at least A/a^2 and scales linearly with the transverse area. In a regular system, for which $A \approx a^2$, we have already verified above that the bound is safe. As we increase the transverse size A at fixed width a, the right hand size of the bound thus grows more rapidly than the entropy S on the left hand side. Hence the bound remains valid for thin systems. With confinement effects included, the ratio $S/(\pi M a)$ actually *decreases* as the transverse size grows: the increase in entropy is more than compensated by larger energy needed to enforce boundary conditions along a bigger surface.

6. Containment energy

In the previous section we argued that confinement—the need to enforce boundary conditions on localized energy eigenmodes—adds energy to systems which naively appear to violate Bekenstein's bound, ensuring that the bound is in fact satisfied. In this section we explore an alternative source of extra mass that has been suggested [27] as a contribution protecting the Bekenstein bound. We will ultimately dismiss this contribution as both insufficient and inessential.

The energy of any quantum state in a cavity generally varies with the volume of the cavity. Indeed, the presence of Casimir energy ensures that this is true even for the vacuum state. This implies that the system exerts pressure on its boundary. The model of the previous section focussed on the need for interactions that keep field modes confined to a given region. It did not explicitly address the requirement that the cavity must be mechanically stable against implosion or expansion, which is clearly necessary if energy eigenstates (which are static) are to be considered. We will now devise a different model for estimating additional energy, which focusses not on interactions but instead on the mechanical properties of a container wall counteracting the pressure.

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As in section 3 and 4 (but in contrast with section 5), we consider free fields obeying sharp boundary conditions. However, we will now take into account the pressure or suction on the boundary surface. We surround the system by a thin wall. This container must be sufficiently rigid to withstand the range of pressures caused by the states one wishes to include in the entropy S. We estimate a lower bound on the mass of the required container. This mass can be expressed as a function of the pressure and the shape of the boundary, generalizing earlier estimates by Bekenstein [27]. Then we investigate how this additional energy affects the validity of Bekenstein's bound in various examples.

6.1 Mass of a thin container

Consider a region $\mathcal{V}$ surrounded by a stationary wall, a container which stabilizes the matter occupying $\mathcal{V}$ against pressure or suction. Let us approximate the container as a codimension-one source of stress-energy, with distributional stress tensor

$$T^{ab} = S^{ab}\delta[r - r_0(\vartheta, \varphi)]\,, \tag{6.1}$$

where r is a gaussian normal coordinate along geodesics generated by the vector field n_a orthogonal to the wall; ϑ and φ are the remaining two spatial coordinates.

The extrinsic curvature of the wall in the spacetime is given by the second fundamental form,

$$K_{ab} = \nabla_a n_b\,. \tag{6.2}$$

The only nonvanishing entries of K_{ab} are the spatial components tangential to the container, which we indicate by indices $i\ldots$ running over ϑ and φ. Let P be an arbitrary point on the wall. By suitable rotation of the coordinate system about n_a, K_{ij} can be diagonalized:

$$K_{ij} = \mathrm{diag}(K_{\vartheta\vartheta}, K_{\varphi\varphi})\,. \tag{6.3}$$

The diagonal components are the inverse curvature radii of the wall in the principal directions at P; negative entries correspond to concave directions.

Let p be the pressure exerted by the enclosed bulk system on the container at the point P. This pressure results in a force $p\,d\vartheta\,d\varphi$ normal to a surface element $d\vartheta\,d\varphi$. The wall exerts a force $S^{ij}K_{ij}\,d\vartheta\,d\varphi$ orthogonal to itself. Thus, the forces cancel if

$$p = -S^{ij}K_{ij}\,. \tag{6.4}$$

Here we are given a problem where p and K_{ij} are specified, and the task is to find the most energy-saving material by which the system can be enclosed and stabilized. That is, we must find a tensor S^{ij} that satisfies eq. (6.4) while minimizing the surface energy density, S^{00}. The latter is constrained by the dominant energy condition [42], which implies for each (fixed) spatial index i that

$$S^{00} \geq \sqrt{(S^{ii})^2 + \sum_{j\neq i}(S^{ij})^2 + (S^{0j})^2}\,. \tag{6.5}$$

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Recall that the container is taken to be a thin wall, i.e., $S^{ab}n_a = 0$. In the frame where K_{ij} is diagonal, the off-diagonal components of S^{ab} do not contribute to the pressure balance in eq. (6.4). Moreover, by eq. (6.5), turning on any off-diagonal components can only increase the minimum energy density of the container. Hence S^{00} is minimized by choosing a stress-free, isotropic[12] wall material:

$$|S^{ij}| = \frac{|p|}{\hat{K}}\,\delta^{ij}\,, \tag{6.6}$$

where

$$\hat{K} = \mathrm{tr}\,|K_{ij}|\,. \tag{6.7}$$

Then we find for the energy per surface area of the container:

$$S^{00} \geq \frac{|p|}{\hat{K}}\,. \tag{6.8}$$

Physically, this means that the surface density is at least of order the pressure times the smallest curvature radius at each point. This lower bound is both necessary and sufficient for the dominant energy condition, so it cannot be improved. The total container mass therefore obeys

$$E_{\mathrm{cont}} = \int d^2x\sqrt{h}S^{00} \geq \int d^2x\sqrt{h}\frac{|p|}{\hat{K}}\,. \tag{6.9}$$

6.2 Examples

Having established a lower bound on the container mass, we may consider the effect of this additional term on the validity of the generalized Bekenstein bound. Thus, let us refine our definition of energy to include the pressure term,

$$E \equiv E_\phi + E_{\mathrm{C}} + E_{\mathrm{cont}}\,. \tag{6.10}$$

We will begin by reconsidering examples in which the bound was earlier found to be satisfied (section 3); then we will return to the problematic examples of section 4.

6.2.1 High-temperature limit and saturation limit

In section 3.1 we considered Q scalar fields in a cavity at temperature $T \gg 1/a$. For simplicity, let us now assume that the cavity is evenly shaped, say spherical with radius R. We work up to factors of order unity. The bulk pressure is

$$p \approx QT^4\,, \tag{6.11}$$

and the trace of the extrinsic curvature is

$$\hat{K} \approx \frac{1}{R}\,. \tag{6.12}$$

[12]More precisely, we take all of the diagonal components to be mutually equal in magnitude. The sign of each S^{ii} must be chosen opposite to the sign of the corresponding K_{ii}. If the signs vary, the material will not be isotropic.

Thus, the mass of the containing wall obeys

$$E_{\rm cont} \gtrsim QR^3T^4 \tag{6.13}$$

by eq. (6.9).

Comparison with eq. (3.3) shows that the wall mass is at least of the same order as the field energy it contains. Unlike the confinement energy, the containment contribution does not become negligible in the high-temperature limit. At all temperatures, its inclusion significantly increases the margin by which the bound is satisfied.

This is true also in the saturation limit, which can be approximately treated within the above argument, by reducing the temperature to $T \approx 1/R$. Instead of approaching saturation to within a factor of 10 or 100, the gap becomes correspondingly larger upon inclusion of $E_{\rm cont}$.

6.2.2 Casimir problem

If it turned out that containment energy necessarily renders the total energy of the system positive, i.e., if

$$E_{\rm cont} \geq |E_{\rm C}| \,, \tag{6.14}$$

then one could argue that the Casimir problem is resolved. In fact, however, there is a simple example in which this cannot be shown.

Consider a spherical cavity containing some field(s) with negative Casimir energy (for example, a massless scalar obeying Neumann boundary conditions [40]). In the vacuum, the bulk pressure is

$$|p| = \frac{|E_{\rm C}|}{4\pi R^3} \,. \tag{6.15}$$

The extrinsic curvature is

$$\hat{K} = \frac{2}{R} \,, \tag{6.16}$$

so that the surface density of the containing wall obeys

$$S^{00} \geq \frac{|E_{\rm C}|}{8\pi R^2} \,. \tag{6.17}$$

Hence, the wall mass satisfies merely

$$E_{\rm cont} \geq \frac{|E_{\rm C}|}{2} \,. \tag{6.18}$$

For fields with $E_{\rm C} < 0$, we are thus unable to conclude that $E_{\rm C} + E_{\rm cont} \geq 0$.

Of course, eq. (6.18) is not generic; it depends on our specific choice of a spherical container, which minimizes wall energy. For many other shapes, containment energy does compensate the Casimir energy. (For example, for a long cylinder one finds that $E_{\rm cont} \geq 2|E_{\rm C}|$.) To realize $E_{\rm cont} + |E_{\rm C}| < 0$, one would require extremely stiff layers ($S^{ii} > S^{00}/2 > 0$), which—though allowed by the dominant energy condition—may not actually occur in nature. Nor should eq. (6.18) be interpreted as a claim that a complete stationary system could have negative total energy due to a Casimir contribution.

The point here is merely that containment is not enough to demonstrate positive total energy; hence, it does not by itself resolve the Casimir problem of the Bekenstein bound.

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6.2.3 Species problem

Bekenstein [27, 28] has argued that containment resolves the species problem as follows. Each species confined to the region $\mathcal{V}$ contributes a Casimir energy $E_{\mathrm{C},q}$, whose pressure or suction must be compensated by a container wall. The wall energy will thus depend roughly linearly on the number of species, whereas the entropy grows only logarithmically. At the large species numbers necessary for apparent violations of the bound, the wall energy will be overwhelming.

This argument is not fully satisfactory. Since the Casimir energy has no definite sign, one could chose a combination of fields whose Casimir energies mostly cancel, yielding only a small correction to the energy of the first single partice state. Then the species problem would arise as in section 4.1. Though containment energy does resolve the species problem in generic examples, this potential loophole remains.

6.2.4 Transverse problem

Returning to the rectangular cavities studied earlier, we note that their boundary surfaces are flat, so that $S^{00} \to \infty$ by eq. (6.8). However, the wall energy can be rendered finite by allowing the boundary to bulge slightly, as long as this does not significantly alter the shape of the cavity and invalidate the analysis of field modes. In particular, the width a of the system should still be approximately a.

Consider, for example, the two large, approximately flat, rectangular surfaces of area bc containing the pancake-shaped system of section 4.3.2. The pressure or suction on this surface is at least $|p| = |\eta|/a^4$ (from the Casimir pressure of the vacuum state). We may assume that $b \equiv a/\delta \leq c \equiv a/\epsilon$. A simple geometric argument shows that the principal curvature radii of this surface must vastly exceed a/δ^2 and a/ϵ^2, respectively, if the surface is to bulge by an amount much less than a. Thus we find that $\hat{K} \ll \delta^2/a$, and by eq. (6.8), the surface energy density must be much larger than $|\eta|/(\delta^2 a^3)$. Hence the total mass of the container must be much larger than $|\eta|/(\delta^3 \epsilon a)$.

Note that this contribution to the total energy scales like a power of the transverse area, and dominates over E_{C} and E_ϕ for thin systems ($\delta \ll 1$), in the regime where the transverse problem arose ($E_\phi \approx 1/a$). The corresponding growth of the bound with transverse area outpaces the logarithmic divergence of the entropy, removing the problem noted in section 4.3.2.

As an example of a pencil-shaped system, consider a cylindrical cavity of length $l \gg R$. It has total curvature $\hat{K} = 1/R$ on its long sides. Because of their much smaller area, we neglect the two disks that complete the boundary. In the ground state the cylinder wall sustains pressure $|p| = |E_{\mathrm{C}}|/\pi R^2 l$. Hence the wall must have a density $S^{00} \geq |E_{\mathrm{C}}|/\pi R l$, and its total mass obeys

$$E_{\mathrm{cont}} \geq 2|E_{\mathrm{C}}|\,. \tag{6.19}$$

This implies that $E_{\mathrm{cont}} + E_{\mathrm{C}}$, the total energy in the ground state, is necessarily positive. Though the energy grows only linearly with transverse area in this case, the container mass bolsters the bound sufficiently to stay ahead of the logarithmic growth of the entropy.

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However, a basic problem of the containment argument is that it relies on the assumption that pressure on the container builds up at least in proportion to the entropy. As pointed out in section 6.2.3, this can be circumvented by cancelling off Casimir energies of different signs. More general problems arise if we do not insist that the walls must be able to contain *all* states up to energy M, but count only those states toward S whose pressure is compatible with the wall design.

For example, the Casimir force on a pair of conducting plates made arbitrarily large by increasing the transverse area. This force can then be balanced by any one of a large number of excited states (since only the wavelength along the short direction is fixed by the cancellation requirement, leaving the transverse wavelength variable). Thus it would seem that a container with little mass could hold a large entropy.

Of course, this is false—but not because of pressure effects. The most general reason why one cannot construct massless capacitor plates is that the necessary charge carriers cost energy. But this is precisely the confinement energy estimated in section 5.

7. Bound states as complete systems

In this section, we summarize the lessons learned from our analysis of examples. We conclude that interactions are crucial for the operation of the Bekenstein bound: without them, the bound is trivial; but with interactions properly included from the start, complete systems simply correspond to bound states, whose number is measured by the entropy S.

The analysis of section 6 has shown that realistic containers add enough mass to protect Bekenstein's bound against many problematic examples of the type constructed in section 4. However, we have found that this is not guaranteed in general, and we have pointed out some loopholes. Confinement energy, as estimated in section 5, appears to provide a more reliable mechanism upholding the bound.

Demanding containment is not only insufficient—at present it is also poorly motivated. Containers did seem necessary in the context in which Bekenstein's bound first arose: systems which are slowly lowered towards a black hole, hovering just above the horizon before being dropped in, are necessarily subjected to accelerations. It is therefore difficult to analyze the Geroch process independently of the specific composition of the system, unless it is placed in a rigid container for the duration of the gedankenexperiment.

However, in the light of recent developments [18, 19, 17, 20, 16, 15], we take the point of view that the Bekenstein bound, like the GSL, should be properly regarded as a consequence of the covariant bound. Then conditions arising in the Geroch lowering process are not necessarily general conditions for the validity of the Bekenstein bound. In the recent derivation of the Bekenstein bound from the GCEB [15], the system follows a geodesic in flat space. Hence, no separate container is required, as long as the system holds itself together through interactions.

We conclude that confinement energy, arising from the need for interactions, is both the more plausible and the more effective correction to consider when analysing apparent violations of Bekenstein's bound: Firstly, it is well-motivated, since the energy eigenstates of free fields cannot be localized to a finite region. Secondly, it is successful, in the sense that the background field model studied in section 5 suggests that confinement contributions invalidate the apparent counterexamples we presented in section 4, without rendering the bound trivial.

Unfortunately, the background field model turns the application of Bekenstein's bound into a two-step process: First, one works out the entropy of an active free field with fixed boundary conditions; then, one estimates the energy of an additional passive field needed for enforcing those boundary conditions. This is ugly, and moreover, both steps are unphysical: the first, because truly free fields cannot obey boundary conditions; the second, because there is no such thing as a classical static background potential.

Thus, the background field model is, at best, a crude prescription for completing incomplete systems. This may be a sensible procedure if one is forced to discuss an apparent counterexample that fails to include all essential parts of a physical object. But it does not itself constitute a satisfactory formulation of the Bekenstein bound, and it cannot be the last word if the bound is to be ascribed any fundamental significance.

However, the success of the background field model does suggest that if Bekenstein's bound has a precise formulation, interactions are of the essence. Instead of patching up fake counterexamples, it would be preferable to eliminate incomplete systems from the start. One would like to restrict the application of the bound to systems which *confine themselves* through interactions, such as a nucleus, an atom, or a crystal. This forces us to abandon the simple model of a finite system as a set of free fields confined by external boundary conditions. Instead, we must consider the proper physical description in terms of interacting fields with no external boundary conditions, such as QCD, QED, the standard model, or some low energy lagrangians obtained from string theory.

Moreover, a statement of the bound with any claim to generality should not require the specification of a particular macroscopic system (e.g., a block of iron) before microscopic states are counted. Instead, it should define the entropy S directly in terms of states in the Fock space of a relevant quantum field theory.

Therefore we propose that complete systems should be formally characterized as *bound states*, i.e., eigenstates of the full interacting hamiltonian which are discrete (have no continuous quantum numbers). The entropy S will still be defined as the logarithm of the number of orthogonal quantum states satisfying macroscopic conditions specified on the right hand side of the bound. But we propose to discard scattering states and take S to count only bound energy eigenstates.

Of course, any bound state gives rise to a continuous family of eigenstates related by overall boosts. In order to mod out by this trivial continuum, one may fix the total three-momentum to take some arbitrary but fixed value. The choice $\mathbf{P} = 0$, the rest frame, has been implicit throughout this paper and in the literature. Then the energy eigenvalue of each bound state is simply its rest mass.

Now one may formulate the Bekenstein bound as follows. Pick upper limits on the rest mass, M, and on the width, a. Define $\mathcal{N}(M, a)$ as the number of bound energy eigenstates with vanishing spatial momentum, eigenvalue $E \leq M$, and support over a region of width a. With

$$S(M, a) \equiv \log \mathcal{N}(M, a)\,, \tag{7.1}$$

the bound takes the form

$$S(M,a) \leq \pi M a\,. \tag{7.2}$$

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References

[1] J.D. Bekenstein, *A universal upper bound on the entropy to energy ratio for bounded systems*, *Phys. Rev.* **D 23** (1981) 287.

[2] J.D. Bekenstein, *Generalized second law of thermodynamics in black hole physics*, *Phys. Rev.* **D 9** (1974) 3292.

[3] R. Bousso, *Bekenstein bounds in de Sitter and flat space*, *J. High Energy Phys.* **04** (2001) 035 [hep-th/0012052].

[4] D. Kutasov and F. Larsen, *Partition sums and entropy bounds in weakly coupled CFT*, *J. High Energy Phys.* **01** (2001) 001 [hep-th/0009244].

[5] J.D. Bekenstein, *Black holes and the second law*, *Nuovo Cim. Lett.* **4** (1972) 737–740.

[6] J.D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D 7** (1973) 2333.

[7] R. Bousso, *The holographic principle*, *Rev. Mod. Phys.* **74** (2002) 825 [hep-th/0203101].

[8] W.G. Unruh and R.M. Wald, *Acceleration radiation and generalized second law of thermodynamics*, *Phys. Rev.* **D 25** (1982) 942.

[9] W.G. Unruh and R.M. Wald, *Entropy bounds, acceleration radiation and the generalized second law*, *Phys. Rev.* **D 27** (1983) 2271.

[10] M.A. Pelath and R.M. Wald, *Comment on entropy bounds and the generalized second law*, *Phys. Rev.* **D 60** (1999) 104009 [gr-qc/9901032].

[11] J.D. Bekenstein, *Non-archimedean character of quantum buoyancy and the generalized second law of thermodynamics*, *Phys. Rev.* **D 60** (1999) 124010 [gr-qc/9906058].

[12] D. Marolf and R. Sorkin, *Perfect mirrors and the self-accelerating box paradox*, *Phys. Rev.* **D 66** (2002) 104004 [hep-th/0201255].

[13] R.M. Wald, *The thermodynamics of black holes*, *Living Rev. Rel.* **4** (2001) 6 [gr-qc/9912119].

[14] J.D. Bekenstein, *Limitations on quantum information from black hole physics*, *Acta Phys. Polon.* **B32** (2001) 3555 [quant-ph/0110005].

[15] R. Bousso, *Light-sheets and Bekenstein's bound*, *Phys. Rev. Lett.* **90** (2003) 121302 [hep-th/0210295].

[16] E.E. Flanagan, D. Marolf and R.M. Wald, *Proof of classical versions of the Bousso entropy bound and of the generalized second law*, *Phys. Rev.* **D 62** (2000) 084035 [hep-th/9908070].

[17] R. Bousso, *A covariant entropy conjecture*, *J. High Energy Phys.* **07** (1999) 004 [hep-th/9905177].

[18] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026.

[19] L. Susskind, *The world as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[20] R. Bousso, *Holography in general space-times*, *J. High Energy Phys.* **06** (1999) 028 [hep-th/9906022].

[21] R. Bousso, E.E. Flanagan and D. Marolf, *Simple sufficient conditions for the generalized covariant entropy bound*, *Phys. Rev.* **D 68** (2003) 064001 [hep-th/0305149].

[22] W. Fischler and L. Susskind, *Holography and cosmology*, hep-th/9806039.

[23] N. Kaloper and A.D. Linde, *Cosmology vs. holography*, *Phys. Rev.* **D 60** (1999) 103509 [hep-th/9904120].

[24] S.D. Unwin, *Possible violations of the entropy to energy ratio bound*, *Phys. Rev.* **D 26** (1982) 944.

[25] D.N. Page, *Comment on a universal upper bound on the entropy to energy ratio for bounded systems*, *Phys. Rev.* **D 26** (1982) 947.

[26] D. Deutsch: *Is there a fundamental bound on the rate at which information can be processed?*, *Phys. Rev. Lett.* **48** (1982) 286.

[27] J.D. Bekenstein, *Specific entropy and the sign of the energy*, *Phys. Rev.* **D 26** (1982) 950.

[28] J.D. Bekenstein, *Entropy bounds and the second law for black holes*, *Phys. Rev.* **D 27** (1983) 2262.

[29] J.D. Bekenstein, *Entropy content and information flow in systems with limited energy*, *Phys. Rev.* **D 30** (1984) 1669.

[30] M. Schiffer and J.D. Bekenstein, *Proof of the quantum bound on specific entropy for free fields*, *Phys. Rev.* **D 39** (1989) 1109.

[31] M. Schiffer and J.D. Bekenstein, *Do zero frequency modes contribute to the entropy?*, *Phys. Rev.* **D 42** (1990) 3598.

[32] J. D. Bekenstein and M. Schiffer: *Quantum limitations on the storage and transmission of information*, *Int. J. Mod. Phys.* **C 1** (1990) 355.

[33] J.D. Bekenstein, *Entropy bounds and black hole remnants*, *Phys. Rev.* **D 49** (1994) 1912 [gr-qc/9307035].

[34] J.D. Bekenstein, *Do we understand black hole entropy?*, gr-qc/9409015.

[35] D.N. Page, *Huge violations of Bekenstein's entropy bound*, gr-qc/0005111.

[36] D.N. Page, *Subsystem entropy exceeding Bekenstein's bound*, hep-th/0007237.

[37] D.N. Page, *Defining entropy bounds*, hep-th/0007238.

[38] J.D. Bekenstein, *On page's examples challenging the entropy bound*, gr-qc/0006003.

[39] R. Bousso and E. Flanagan, in preparation.

[40] M. Bordag, U. Mohideen and V.M. Mostepanenko, *New developments in the Casimir effect*, *Phys. Rept.* **353** (2001) 1 [quant-ph/0106045].

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[41] L. Susskind and J. Uglum, *Black hole entropy in canonical quantum gravity and superstring theory*, *Phys. Rev.* **D 50** (1994) 2700 [hep-th/9401070].

[42] R.M. Wald: *General relativity* the University of Chicago Press, Chicago 1984.